Relative potency plays a key role in understanding the relationship between the doses of two treatments. Defined as the ratio of equally effective doses, it is central to communicating the relationship between a new drug entering the market and older medications. Because doses are tested and collected at discrete points in clinical trials, estimation in direct assays is hampered by the fact that clinical responses and tolerances are usually only available as grouped data which is frequently left- or right-censored information or both. This problem was considered in Bonzo and Laska (2004), which modeled premature discontinuation of treatment. However, further difficulties arise when the risk/benefit assessment is multivariate in dimension. This paper proposes an extension of the author’s previous work for estimating relative potency in the multivariate setting utilizing dimension reduction and generalized mixed data model (GMDM) approaches. Large sample approximation and an analogue of Fieller’s theorem are used to construct confidence interval estimates. Simulated data and data from a clinical trial are used to illustrate the methods’ use.

Key Words: left-/right-censored information, grouped data, generalized mixed data model, fiducial limits

1. Introduction

Direct assays have been used as one of the primary methods in drug formulation. They are characterized by the application of doses of a test and standard drug formulations (stimuli) to subjects (usually humans) to produce a directly measurable response. The dose that produces the desired response is called the subject’s tolerance dose. Central to their analysis is the estimation of relative potency, a ratio of two equally effective doses. As used in pharmacological studies, relative potency helps gauge how the recommended dose of a test drug is formulated using an established drug as a base comparator. In switching studies, it is used to communicate the relationship between a new drug entering the market and older medications in the same pharmacologic class.

The construct for estimating relative analgesic potency using the direct assay approach when the only available information is the dose interval in which the drug’s desired effect took place is as follows (Bonzo and Laska, 2004). Let \( Y_1, \ldots, Y_n \) be IID with distribution \( F \) and consider fixed points \( a_1, \ldots, a_k \) in the support of \( F \) such that \( a_{i-1} < a_i \), \( i = 1, \ldots, k \), where \( a_0 = -\infty \) and \( a_k = +\infty \). We then use the \( a_i \)'s to define intervals \((a_{i-1}, a_i]\) and let \( X_i \) represent the count of \( Y_i \)'s in \((a_{i-1}, a_i]\), \( i = 1, \ldots, k \). Note that the count for the right-censored interval \((a_{k-1}, \infty)\) is assumed to contain non-responders.

If \( Y_1, \ldots, Y_n \) is IID from \( F(y; \theta) \) where \( \theta \in \Theta \subset R^k \) and the \( Y_i \)'s are observed at intervals defined by fixed points \( a_1, \ldots, a_k \) such that \( a_{i-1} < a_i \), \( i = 1, \ldots, k \), where \( a_0 = -\infty \) and \( a_k = +\infty \). Estimation of the unknown parameter \( \theta \) can be done by maximizing the likelihood

\[
L(\theta) = \prod_{i=1}^{k} (F(a_i; \theta) - F(a_{i-1}; \theta))^{X_i},
\]

or equivalently by maximizing the log-likelihood given by

\[
l(\theta) = \sum_{i=1}^{k} X_i \log [F(a_i; \theta) - F(a_{i-1}; \theta)].
\]

The value of the maximum likelihood estimator (MLE) \( \hat{\theta} \), can be extracted using an iterative approach such as the Newton-Raphson method. At the \( (m+1) \)th iteration the estimate is given by
\[ \theta^{(m+1)} = \theta^{(m)} - H^{-1}(\theta)S(\theta) \bigg|_{\theta = \theta^{(m)}} \]

where \( S(\theta) = \frac{\partial}{\partial \theta} l(\theta) \) is the Fisher score function and \( H(\theta) = \frac{\partial^2}{\partial \theta \partial \theta'} l(\theta) \) is the Hessian matrix. Iteration is terminated when \( |\theta^{(m+1)} - \theta^{(m)}| \leq \epsilon \), for some pre-specified tolerance \( \epsilon \).

Since \( \hat{\theta} \) is MLE, it follows that \( \hat{\theta} \xrightarrow{d} \mathcal{N}(\theta, I^{-1}(\theta)) \). Hence the standard error of \( \hat{\theta} \) is given by \( \text{se}(\hat{\theta}) = \text{diag}(I^{-1/2}(\theta)) \) and the approximate 95% confidence interval for \( \theta_i \) is given by \( \hat{\theta}_i \pm 1.96 \text{se}(\hat{\theta}_i) \).

The paper is organized as follows. Section 1 discusses the general estimation approach for relative potency using interval information. Section 2 presents the multivariate extension of the problem utilizing dimension reduction and generalized mixed data model (GMDM) approaches. Section 3 shows the large sample approximation results for the inference of relative potency in the multivariate setting, as well as an analogue of Fieller’s theorem. Application of the method using simulated data is presented in Section 4. Finally, Section 5 gives some concluding remarks.

2. Multivariate Extension

Consider a set of mixed data observations \( \{(u_j, y_{2j}, y_{3j})_j, j = 1, \ldots, M\} \). Let \( \{(y_{ij1}, \ldots, y_{ijn})_i, i = 1, \ldots, N\} \), denote the state-induced data where the 1x(1+d_2+d_3) vector \( y_{ij} \) denotes the \( j \)th data observation in the \( i \)th state. Alternatively, we can represent the same data as \( \{(y_{ij1}, y_{2ij}, y_{3ij}), j = 1, \ldots, n_i; i = 1, \ldots, N\} \) where \( y_{ij1} \) is equal to 1; \( y_{2ij} \) is a 1xd_2 vector of continuous observations; and \( y_{3ij} \) is a 1xd_3 vector of ordinal observations.

We assume that each observation \( y_{ij} \) in the \( i \)th state follow some generalized linear model (GLIM) specification given by \( \eta_{ij} = h(E(y_{ij})) \) for some natural link function \( h \) where the natural parameter \( \eta_{ij} \) is linear with respect to a px(1+d_2+d_3) vector of common regression coefficients \( \beta \), i.e., \( \eta_{ij} = x_{ij} \beta \) for some 1xp vector of regressors \( x_{ij} \). Instead of working directly with the GLIM specification above to make an inference on \( \beta \), we follow the conditioning approach used by Fitzmaurice and Laird (1995) and extend it to include the ordinal variables. This approach structurally conforms with the generalized mixed data model (GMDM) introduced by de Leon and Carriere (2007) without the burden of specifying the actual joint distribution of the vector of mixed random variables.

We assume that the GLIM specification is equivalent to the conditional specification given by

\[ \eta_{ij1} = h_1(E(y_{ij1})) = x_{ij} \beta_{(1)} \]

\[ \eta_{2ij1} = h_2(E(y_{2ij1} | y_{ij1})) = x_{ij} \beta_{(2)} + (y_{ij1} - \eta_{ij1}) y_{1} \]

and

\[ \eta_{3ij21} *= h_3(E(y_{3ij21} | y_{2ij1}, y_{ij1})) = x_{ij} \beta_{(3)} + (y_{2ij1} - \eta_{2ij1}) y_{2} + (y_{ij1} - \eta_{ij1}) y_{3} \]

where \( \{y_{3ij*}, j = 1, \ldots, n_i; i = 1, \ldots, N\} \) is a set of continuous latent variables related to the ordinal measurements \( \{y_{3ij}, j = 1, \ldots, n_i; i = 1, \ldots, N\} \).

Following de Leon and Carriere (2007), we assume that the latent relationship between \( y_{3ij*} \) and \( y_{3ij} \) given \( y_{2ij} \) and \( y_{ij1} \) is defined by the threshold model:
\[ y_{3dij} = l \leftrightarrow \Delta_{d,j-1} < y_{3dij} \leq \Delta_{d,j} \]

where \( y_{3dij} \) and \( y_{3dij} \) are the \( d \)th components of \( y_{3ij} \) and \( y_{3ij} \), respectively, \( d = 1, \ldots, d \); \( \eta_{3dij} \) is the \( d \)th component of \( \eta_{3ij} \); \( l = 1, \ldots, m_d \) are the ordinal scores for \( y_{3dij} \); and \( \{ \Delta_{d,0} = -\infty, \Delta_{d,1}, \ldots, \Delta_{d,m_d-1}, \Delta_{d,m_d} = +\infty \} \) are the unknown cutpoints.

To facilitate the estimation of the regression coefficients \( \beta \), we propose the use of the generalized estimating equation (GEE) approach by suitably assuming working covariances for \( y_{ij} \), \( y_{2ij} \) given \( y_{ij} \) and \( y_{3ij} \) given \( y_{2ij} \) and \( y_{ij} \). We let \( \text{cov}(y_{ij}) = v_{ij} \approx \sigma_i^2 \); \( \text{cov}(y_{2ij} | y_{ij}) = v_{2i} \approx \Sigma_{2i} \), and \( \text{cov}(y_{3ij} | y_{2ij}, y_{ij}) = v_{3i} \approx \Sigma_{3i} \).

Given the specification in (2.1), let \( \alpha_1 = \beta_{(1)} \), \( \alpha_2 = (\text{vec}(\beta_{(2)})', \text{vec}(\gamma_1')')' \), and \( \alpha_3 = (\text{vec}(\beta_{(3)})', \text{vec}(\gamma_2')', \text{vec}(\gamma_3')')' \) where vec(A) stacks the column of any \( m \times n \) matrix A into an \( mn \times 1 \) vector. Similarly, define \( w_{ij} = x_j \); \( w_{2ij} = (I_{d_2} \otimes x_j, I_{d_2} \otimes (y_{ij} - \eta_{ij})) \); and \( w_{3ij} = (I_{d_3} \otimes x_j, I_{d_3} \otimes (y_{2ij} - \eta_{2ij}), I_{d_3} \otimes (y_{ij} - \eta_{ij})) \), where \( \otimes \) denotes the Kronecker product.

This re-definition, gives an alternative form to (2.1) given by \( \eta_{ij} = w_{ij} \alpha_1 \); \( \text{vec}(\eta_{2ij}) = w_{2ij} \alpha_2 \); and \( \text{vec}(\eta_{3ij}) = w_{3ij} \alpha_3 \). The set of GEE’s for \( \alpha = (\alpha_1', \alpha_2', \alpha_3', \gamma_1', \gamma_2', \gamma_3')' \) is given by the set of \( p(1+d_2+d_3)+d_1+d_2d_3+d_3 \) equations

\[
\sum_{i=1}^{N} \sum_{j=1}^{n_i} u_i (\alpha, \Sigma, \Delta) = \sum_{i=1}^{N} \sum_{j=1}^{n_i} z_i (\alpha)' d_i (\alpha, \Delta) v_i (\Delta, \alpha) j = 0 \quad (2.2)
\]

where \( z_i (\alpha) \) is a \( (1 + d_2 + d_3) \times p(1 + d_2 + d_3) + d_1 + d_2d_3 + d_3 \) matrix of constants; \( v_i (\Sigma, \Delta) \) is the matrix of covariances; \( r_i (\alpha, \Delta) \) is the vector of errors; \( d_i (\alpha, \Delta) \) is a \( (1 + d_2 + d_3) \times (1 + d_2 + d_3) \) block-diagonal matrix; and \( \Delta \) represents the set of unknown cutpoints \( \{ \Delta_{d,j}, l = 1, \ldots, m_d - 1; d = 1, \ldots, d_3 \} \).

### 3. Estimation of Relative Potency Using the Direct Assay Approach

Consider the problem of a multivariate parallel-line bioassay in which the outcomes of interest are of a mixed data structure. Suppose we stratify line bioassay in which the outcomes of interest are of a mixed data structure. Suppose we stratify line bioassay into an estimated estimating equation (GEE) approach by suitably assuming working covariances between standard and test drugs coefficients and \( x \).

This condition of similarity can be broken down into two hypotheses, namely:

\[
\eta_i = \beta_{11} + \beta_{12} x_i, \quad i = 1, \ldots, T_i \quad \text{and} \quad \eta_{is} = \beta_{20} + \beta_{21} x_{is}, \quad s = 1, \ldots, S_i
\]

where \( \eta_i = h(E(y_i)) \) is a natural link function, \( \beta_{ij} ' \)’s are \( 1 \times (1+d_2+d_3) \) vector of regression coefficients and \( x_i \) represents log dose levels. The fundamental condition of similarity between standard and test drugs in (3.1) requires that \( \eta_i = \eta_{is} \) when \( x_{it} = \mu + x_{is} \), \( \mu = \log \rho \), \( \rho \) being a measure of common relative potency of the test drug relative to the standard drug.
\[ (i) \quad H_m : \beta_{11} = \beta_{21} = \beta_1 \quad \text{and} \quad (ii) \quad H_s : \beta_{10} = \beta_{20} + \beta_1 \mu. \]

\( H_m \) is called the hypothesis of marginal parallelism while \( H_s \) is called the hypothesis of (main) similarity condition. These two hypotheses are tested sequentially with \( H_m \) tested first and then \( H_s \) next if \( H_m \) is rejected.

Clearly, the set of equations in (3.1) can be represented in multivariate regression form
\[ \eta_i = X_i \beta. \quad (3.2) \]

Similarly, the fundamental condition of similarity can be represented in terms of a general linear hypothesis form with (i) given by \( H_m : C_m \beta = 0 \) and (ii) given by
\[ H_s : C_s(\mu)\beta = 0 \quad \text{where} \quad C_m = \begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad C_s(\mu) = \begin{pmatrix} 1 - \mu & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \]

Our suggested approach to treating this problem when data is of the mixed type proceeds from utilizing the \( X \) matrix structure given in (3.2) for use in a conditional model given by (2.1).

Partitioning \( \beta \) as \( \beta = (\beta_{(1)}, \beta_{(2)}, \beta_{(3)}) \) where \( \beta_{(j)} = (\beta_{1(j)}, \beta_{1(j)}', \beta_{2(j)}, \beta_{2(j)}')', j = 1, 2, 3 \) and \( \beta_{(1)}, \beta_{(2)} \) and \( \beta_{(3)} \) represent the coefficients for the nominal, continuous and ordinal outcomes, respectively. Based on this partition, the hypothesis of marginal parallelism is given by \( H_m : C \beta^* = 0 \) where
\[ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{d_2} & 0 \\ 0 & 0 & I_{d_5} \end{pmatrix} \otimes C_m. \]

Using the asymptotic result for \( \hat{\beta}^* \), \( H_m \) can be tested using the Wald-type chi-square given by
\[ \chi^2_w = (C \hat{\beta}^*)' (\hat{H}_1^{-1} \hat{G}_1^{-1} \hat{H}_1^{-1})^{-1} (C \hat{\beta}^*) \]
where \( \hat{H}_1 \) and \( \hat{G}_1 \) are obtained from the alternative representation of (2.1) using the \( X \) matrix in (3.2) and \( \hat{\beta}^* \) is the corresponding GEE estimator obtained from (2.2). Under \( H_m \),
\[ \chi^2_w \rightarrow \chi^2_{1+d_2+d_5}. \]

Testing the hypothesis of similarity is equivalent to testing \( H_s : C(\mu)\beta^* = 0 \) where
\[ C(\mu) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{d_2} & 0 \\ 0 & 0 & I_{d_5} \end{pmatrix} \otimes C_s(\mu) \quad (3.3) \]
for some constant \( \mu = \log \rho \), \( \rho \) being a measure of common relative potency. Using the same approach as that for \( H_m \), \( H_s \) can be tested using the Wald-type chi-square given by
\[ \chi^2_w = (C(\mu)\hat{\beta}^*)' (\hat{H}_1^{-1} \hat{G}_1^{-1} \hat{H}_1^{-1})^{-1} (C(\mu)\hat{\beta}^*) \]
where \( \hat{H}_1 \), \( \hat{G}_1 \) are obtained from the alternative representation of (2.1) using the \( X \) matrix in (3.2) and \( \hat{\beta}^* \) is the corresponding GEE estimator obtained from (2.2). Under \( H_s \),
\[ \chi^2_w \rightarrow \chi^2_{2(1+d_2+d_5)}. \]
If we let $a = (a_1, a_{21}, ..., a_{2d}, a_{31}, ..., a_{3d})'$ and $b = (b_1, b_{21}, ..., b_{2d}, b_{31}, ..., b_{3d})'$ where $a_y = \hat{\beta}_{10(y)} - \hat{\beta}_{20(y)}$ and $b_y = \hat{\beta}_{110(y)}$, respectively. Also, let $F = \left( \hat{H}_1^{-1} G \hat{H}_1^{-1} \right)^{-1} = (f_y)$. Assuming that $H_m$ and $H_s$ are not rejected, an asymptotic Fieller-type fiducial limits for $\mu$ can be obtained by solving the inequality

$$(a + b\mu)'F(A + b\mu) \leq \chi^2_{a,1+d,1+d}$$

(3.4)

for $\mu$ where $\chi^2_{a,1+d,1+d}$ is the upper $\alpha$ percentage point of the chi-square distribution with $1 + d_2 + d_3$ degrees of freedom. Solving (3.4) yields the limits

$$(\mu_L, \mu_U) = \hat{\mu} \pm \sqrt{\hat{\mu}^2 + \frac{1}{b'Fb} (\chi^2_{a,1+d,1+d} - a'Fa)}$$

where $\hat{\mu} = \frac{a'Fb}{b'Fb}$. Thus, $\frac{a'Fb}{b'Fb}$ is a logical choice for the value of $\mu$ in (3.3) above.

4. Some Simulation Results

The data for this application were simulated from a clinical trial evaluating interval dose information. A total of four levels were used for the study drug (standard), active comparator (test) and placebo. Dose intervals provided here represent the dose range at which subjects attained stable pain. Table 1 shows the interval counts for both the study drug and active comparator. Counts for placebo were not included. Within the intent-to-treat population, a total of 32 subjects were exposed to the active comparator, while 33 subjects were exposed to the study drug. The dose levels used for the active comparator were 20, 40, 60, and 80 dose units. The corresponding dose levels for the study drug were 12.5, 25, 37.5, and 50 dose units.

Table 1. Stable Dose Interval information for Study and Active Comparator Drugs (Intent-to-Treat Population)

<table>
<thead>
<tr>
<th>Interval</th>
<th>Count</th>
<th>Interval</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty,20]$</td>
<td>3</td>
<td>$(-\infty,12.5]$</td>
<td>7</td>
</tr>
<tr>
<td>(20,40]</td>
<td>6</td>
<td>(12.5,25]</td>
<td>3</td>
</tr>
<tr>
<td>(40,60]</td>
<td>11</td>
<td>(25,37.5]</td>
<td>10</td>
</tr>
<tr>
<td>(60,80]</td>
<td>4</td>
<td>(37.5,50]</td>
<td>3</td>
</tr>
<tr>
<td>(80,\infty)</td>
<td>8</td>
<td>(50,\infty)</td>
<td>10</td>
</tr>
<tr>
<td>Total</td>
<td>32</td>
<td>Total</td>
<td>33</td>
</tr>
</tbody>
</table>

Estimation and test results are given in Table 2. Results show that for the active comparator group, the maximum likelihood estimate of the model parameters is given by (0.2219, 44.7935, 383.6393). For the study drug group, the maximum likelihood estimate of the parameters is given by (0.2130, 26.1795, 391.6922).

Hypothesis testing results show that the joint hypothesis given by $H_1$ is rejected in the active comparator group ($p = 0.0102$). However, the same hypothesis was not rejected in the study drug group ($p = 0.01246$). Test results for $H_2$ showed that both groups contain non-responders ($p = 0.0030$ for active comparator and $p = 0.0466$ for study drug). The hypothesis $H_3: \mu = 50$ was not rejected in the active comparator group ($p = 0.2093$), while the hypothesis $H_3: \mu = 30$ was not rejected in the study drug group ($p = 0.4614$). Hence, if
non-responders are excluded, we can view the active comparator and study drug groups as having mean tolerance doses of 45 dose units and 26 dose units, respectively.

<table>
<thead>
<tr>
<th>Statistic/Hypothesis</th>
<th>Active Comparator (N = 32)</th>
<th>Study Drug (N = 33)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha (SE)</td>
<td>0.2219 (0.0749)</td>
<td>0.2130 (0.1070)</td>
</tr>
<tr>
<td>Mean (SE)</td>
<td>44.7935 (4.1464)</td>
<td>26.1795 (5.1868)</td>
</tr>
<tr>
<td>Variance (SE)</td>
<td>383.6393 (151.3384)</td>
<td>391.6922 (234.3714)</td>
</tr>
<tr>
<td>Median</td>
<td>51.9504</td>
<td>33.0270</td>
</tr>
</tbody>
</table>

Hypothesis

- $\alpha = 0$ and $\mu = 50/\mu = 30$
  - Active Comparator: 9.1728 (0.0102)
  - Study Drug: 4.1660 (0.1246)
- $\alpha = 0$
  - Active Comparator: 8.7771 (0.0030)
  - Study Drug: 3.9609 (0.0466)
- $\mu = 50/\mu = 30$
  - Active Comparator: 1.5765 (0.2093)
  - Study Drug: 0.5425 (0.4614)

Relative Potency

(Comparator/Study)

- Relative Potency: 1.5730
- Standard Error: 0.2771
- Delta Method: (1.0298, 2.1161)
- Fieller’s Method: (1.435, 2.3317)

Relative potency calculation yielded an estimate of 1.57. This indicates that 1.57 dose units of the active comparator is equivalent to 1 dose unit of the study drug. The standard error estimate is 0.2771 which indicates an adequate precision level. The delta method yielded an approximate 95% CI given by (1.0298, 2.1161), while Fieller’s method yielded an approximate 95% fiducial limits of (1.435, 2.3317). Examination of the interval widths shows that Fieller’s method yielded a relatively tighter interval compared to the delta method.

5. Conclusions

The preceding section demonstrated that valid relative potency estimates can be formulated using the direct assay approach even if the effective dose information is represented in interval form. In the application demonstrated, stable pain was defined as achievement of a pre-specified pain level within n-1 out of a rolling window of n days. One can argue that the definition, though operational, does not equate to adequate pain control. One can argue that such concerns can be addressed more adequately through an indirect assay where the response can be represented by a mixed-data structure. Analysis of such data types can be done using the extension results presented in Sections 3 and 4.

References

