Estimation the Yield Curve of Nelson-Siegel Model and its Extensions by L-BFGS-B Method Optimization Approach

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Abstracts

This paper discuss yield curve by using Nelson-Siegel and its extensions. The Extensions of Nelson-Siegel models discussed are Nelson-Siegel Model, Svensson Model, Rezende-Ferreira model and Svensson extended model. Svensson model development was begun by some practitioners who were thinking that macro and micro economic factors affected bond yields. The development of this model is done by adding these factors into Svensson model. The model class has linear and nonlinear guidelines so these models have multiple local minima. This condition causes model estimation more difficult to estimate. We estimate these models by using NLS estimation with L-BFGS-B method optimization Approach.

Keywords: Yield Curve, Svensson Extended Model, Macro and Microeconomics

1. Introduction

The yield curve can be modeled based on interest rate period, which is forward rate, discount factor, and zero coupon. In those variety of periods, various models can be used namely Nelson-Siegel Model class. It is a parametric model consisting of three factors, i.e., flat, S, and hump shapes (Nelson and Siegel, 1987).

In following few years, Nelson-Siegel model has multiple developments. Svensson (1994) added a second hump in Nelson-Siegel model so that model became a model of the four factors often called Svensson (SV) model. Rezende and Ferreira (2011), to add accuracy of the model, proposes addition of the third hump into Svensson model. With these additions, Svensson model became a five-factor model or Ferreira-Rezende model (RF).

From these models, determination of yield uses only maturity. According to Rahardjo (2003), bond yields are influenced by economic and non-economic factors in trade market. Diebold, et al (2007) suggested that yield in each country is very different depending on the state of economic factors. This paper is to develop Svensson models accommodating other factors that affect of the yield. Development of this model will will reveal correlation between yield and factors that influence it, and will determine relationship whether it is linear or not. To see the relationship in this model, we use resettest test (Ramsey, 1969).

Nelson-Siegel model class has linear and nonlinear parameters. In these conditions, these models have multiple local minima so that estimation cannot use usual estimation. Previous studies have widely discussed the estimation of Nelson-Siegel model class, they are Bolder and Streliski (1999), Maria, et al (2009), Gilli, et al (2010), Rezende and Ferreira (2011), and Rosadi (2011).

As stated previously, these models have multiple local minima. To overcome this problem in these conditions, we propose NLS estimation with L-BFGS-B method optimization approach. This optimization method is an extension of the limited memory BFGS method (LM-BFGS or L-BFGS) which uses simple boundaries of the model (Zhu et al, 1997). This method can solve the problem of a model that has multiple local minima. BFGS is a compounding of development, which is Broyden, Fletcher, Goldfarb, and Shanno.

2. Nelson-Siegel Model Class

This section will discuss Nelson-Siegel model and its development. Those models are used to describe yield curve. This model is a parametric model that has several factors. Nelson-Siegel model class discussed here are:

Nelson-Siegel model (NS) was first developed by Charles Nelson and Andrew Siegel from the University of Washington in 1987. This modeling is based on various terms to maturity that describe yield curve, such as: flat, hump, and *S* shapes (Nelson and Siegel, 1987). The model is formulated as

$$f_i(\lambda;\ \boldsymbol{\beta},\tau) = \beta_0 + \beta_1\, exp\left(-\frac{\lambda_i}{\tau}\right) + \beta_2 \frac{\lambda_i}{\tau} exp\left(-\frac{\lambda_i}{\tau}\right), \tag{1}$$

where f_i is forward rate of government bond in $i, i = 1,\cdots,p$. i is number of bonds, λ is time to maturity, $\boldsymbol{\beta}$ is linear parameter vector, i.e., $\boldsymbol{\beta} = (\beta_0,\ \beta_1,\ \beta_2)^T$, and τ is a nonlinear parameters of maturity. β_0 is a function of early maturity, be worth staying same if maturity is close to zero, β_1 determines onset curve (short term) in a variety of irregularities, this curve will be negatively skewed if parameters are positive, forward rate function will converge to $\beta_0 + \beta_1$ if τ approaches infinity, β_2 determines magnitude and direction of the hump, and τ determines the position of the hump or *U* shape curve. Svensson (1994) developed NS model by adding a second hump on model to be a four-factor model better known as the model of Svensson (SV). The model is written as.

$$f_i(\lambda;\ \dot{\boldsymbol{\beta}},\boldsymbol{\tau}) = \beta_0 + \beta_1\, exp\left(-\frac{\lambda_i}{\tau_1}\right) + \beta_2\left[\frac{\lambda_i}{\tau_1} exp\left(-\frac{\lambda_i}{\tau_1}\right)\right] + \beta_3\left[\frac{\lambda_i}{\tau_2} exp\left(-\frac{\lambda_i}{\tau_2}\right)\right], \tag{2}$$

f_i is forward rate function of government bonds, which i, $i = 1,\cdots,p$. i is number of bonds, λ is time to maturity, $\dot{\boldsymbol{\beta}}$ is linear parameter vector, i.e. $\dot{\boldsymbol{\beta}} = (\beta_0,\ \beta_1,\ \beta_2,\beta_3)^T$, $\boldsymbol{\tau} = (\tau_1,\tau_2\ )^T$ is also a parameter of maturity. β_0 is a constant value of forward rate function, it is always constant if maturity period is close to zero, β_1 determines initial value of the curve (short-term) in various terms of aberration, the curve will be negatively skewed if parameter is positive and vice versa. β_2 determines magnitude and direction of the hump curve, if β_2 is positive then hump will occur τ_1, if β_2 is negative then *U* shape curve will occur on τ_1, and β_3 aqua to β_2 which determines magnitude and direction of the second hump, τ_1 is first hump special position or shape of the *U* curve, τ_2 is second hump position or shape of the *U* curve. Rezende and Ferreira (2011) added a third hump of SV models to obtain five-factor model (model RF) as follows:

$$\begin{aligned} f_i(\lambda;\ \ddot{\boldsymbol{\beta}},\dot{\boldsymbol{\tau}}) &= \beta_0 + \beta_1\, exp\left(-\frac{\lambda_i}{\tau_1}\right) + \beta_2\left[\frac{\lambda_i}{\tau_1} exp\left(-\frac{\lambda_i}{\tau_1}\right)\right] \\ &+\beta_3\left[\frac{\lambda_i}{\tau_2} exp\left(-\frac{\lambda_i}{\tau_2}\right)\right] + \beta_4\left[\frac{\lambda_i}{\tau_3} exp\left(-\frac{\lambda_i}{\tau_3}\right)\right], \end{aligned} \tag{3}$$

where $f_i(\lambda;\ \ddot{\boldsymbol{\beta}},\dot{\boldsymbol{\tau}})$ is forward rate function, λ is time to maturity, $(\beta_0,\beta_1,\beta_2,\beta_3,\beta_4)^T$, $\dot{\boldsymbol{\tau}} = (\tau_1,\tau_2,\tau_3)^T$, and $i = 1,\cdots,p$.

Based on introduction above, these models only describe yield curve for maturity. It is necessary to further develop macro and micro economic factors that affect yield. The model developed in this paper is a model of Svensson. Of the second model, it is obtained:

$$f_i(\boldsymbol{\Gamma};\boldsymbol{\xi}) = \beta_0 + \beta_1\, exp\left(-\frac{\lambda_i}{\tau_1}\right) + \beta_2\left[\frac{\lambda_i}{\tau_1}\left(exp\left(-\frac{\lambda_i}{\tau_1}\right)\right)\right] +$$

$$\beta_3 \left[\frac{\lambda_i}{\tau_2}\left(exp\left(-\frac{\lambda_i}{\tau_2}\right)\right)\right] + \alpha_1 f_1(\gamma_1, X_1) + \cdots + \alpha_m f_m(\gamma_m, X_m), \qquad (4)$$

f_i is forward rate function of i government bond, where $i = 1, \cdots, p$, λ is time to maturity, $\mathbf{\Gamma} = \left(\widetilde{\boldsymbol{\theta}}, \alpha_j, \gamma_j\right)^T$, $\dot{\boldsymbol{\beta}}$ is linear parameter vector, i.e. $\dot{\boldsymbol{\beta}} = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5)^T$, $\boldsymbol{\tau} = (\tau_1, \tau_2)^T$ is parameter of maturity, $f_1(\gamma_1, X_1) + \cdots + f_m(\gamma_m, X_m)$ is an additional function of the macro and micro economic factors that affect bond yields, $j = 1, 2, \cdots, m$, ξ is a variable that affects yield, and $\widetilde{\boldsymbol{\theta}} = (\dot{\boldsymbol{\beta}}, \boldsymbol{\tau})$.
Before conducting development of a model, there are a few things done to the micro and macro economic factors, namely: Firstly, to test correlation among micro or macro factors on yield. Secondly, to test of linearity among micro or macro factors to yield. After doing both of these, expansion models of Svensson models is formed.

In this case, macro and micro economic factors affecting yield curve are Jakarta Composite Index (JKSE) and foreign exchange dollar to rupiah (Kurs). Rosadi et al. (2008) suggested that macroeconomic factors that may affect government bond yield curve are rising interest rate of Bank Indonesia (SBI), repayment, volatility, liquidity, reinvestment, decline in purchasing power, inflation, changes in policy, as well as other factors such as stability of the state security, foreign exchange, Jakarta Composite index (JKSE), and so on. Eckhold (1998) in his study states that there is influence between exchange rates on bond yields in study period 1988-1997. In period between years, 2006-2011, it showed an association between JKSE and bond yields which is almost like an opposite mirror (Anonymous, 2011). Qomariyah (2010) argues that in 2010, price index rose whiles government bond yields fell. Mid-May 2006 government bond price showed a very sharp decline though bond price rose again in following week. It occurs due to exchange rate, JKSE keeps under pressure, regional sentiment (decreased partial exchange rate countries in Asia region against U.S. dollar), and plan to increase interest rates by Fed (Elfithasari, 2008). In this paper, researchers conduct observations of bond data, Kurs and JKSE on January 3 to May 31, 2010 which shows that yields and exchange rates have a correlation of 0.1092287 with p-value 7.558e-07 and have a relationship which is nonlinear regression p-value = 3.548e-05 and so is relationship between yield and JKSE has a correlation of -0.09771548 with p-value = 9.76e-06 and have a relationship with nonlinear regression p-value = 0.04534.

Many diagnostics and explanations of these observations, researchers develop SV model of the model (1) so following model can be established as:

$$f_t(\mathbf{\Gamma}; \boldsymbol{\xi}) = \beta_0 + \beta_1\, exp\left(-\frac{\lambda_i}{\tau_1}\right) + \beta_2 \left[\frac{\lambda_i}{\tau_1}\left(exp\left(-\frac{\lambda_i}{\tau_1}\right)\right)\right] + \beta_3 \left[\frac{\lambda_i}{\tau_2}\left(exp\left(-\frac{\lambda_i}{\tau_2}\right)\right)\right]$$
$$+\alpha_1 \exp(-\gamma_1 kurs) + \alpha_2\, exp(-\gamma_2 JKSE), \qquad (5)$$

where f_t is forward rate function, $t = 1, \cdots, p$, λ is time to maturity, $\mathbf{\Gamma} =$

$\left(\widetilde{\boldsymbol{\theta}}, \alpha_1, \alpha_2, \gamma_1, \gamma_2\right)^T$, $\dot{\boldsymbol{\beta}}$ is parameters vector, i.e. $\dot{\boldsymbol{\beta}} = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5)^T$, $\boldsymbol{\tau} = (\tau_1, \tau_2)^T$ is parameters vector of maturity. ξ is an independent variable that consists of maturity, Kurs and JKSE.

3. Estimation of Model

The Estimation of Nelson-Siegel model class is done by determining residual function of each class of Nelson-Siegel model. From these residual functions of the model, its Nonlinear Least Square (NLS) is determined. NLS function is optimized by L-BFGS-B method. This method is an extension of L-BFGS or LM-BFGS (limited memory BFGS) method which has a parameter constraints. BFGS method is a combination method proposed by Broyden, Fletcher, Goldfarb, and Shanno. This method is an extension of quasi-Newton method or the method of descent (Griva et al, 2009). As presented by Kelly (1999), residual function is given as:

$$min\, \varphi(\Gamma), \qquad \Gamma \in \Omega \tag{6}$$

with

$$\Gamma \in \Omega = \{\Gamma \in \Re^{\mathrm{N}} | l_k \leq \Gamma_k \leq u_k\}$$

l_k and u_k are respectively an upper limit and lower limit parameters. Γ_k, φ is NLS function of residual functions of Nelson-Siegel model class and Γ is a parameter of the Nelson-Siegel model class. The magnitude of the parameters above is calculated by method of BFGS.

$$B_{k+1} = B_k - \frac{(B_k s_k)(B_k s_k)^T}{s_k^T B_k s_k} + \frac{d_k d_k^T}{d_k^T s_k}. \tag{7}$$

Where B_k is an assessment of inverse Hessian matrix, for $B_1 = I$, $s_k = \Gamma_{k+1} - \Gamma_k = \delta_k p_k$, δ_k is interval length, p_k is a search direction, and $d_k = \nabla\varphi_{k+1}(\Gamma) - \nabla\varphi_k(\boldsymbol{\Gamma})$, $p_k = -B_k \nabla\varphi_k(\Gamma)$.

From this description of the estimation and optimization, following steps can be summarized:

1. Find the residual function (r) of each model,
2. Find NLS estimation, i.e. $\varphi = \frac{1}{2}\sum_{i=1}^{p}[r_i]^2$, of each model,
3. Find the $p \times p$ matrix value for $B_1 = I$, p is the number of parameters estimated in each model,
4. Find the initial value of parameter vector with rank $p \times 1$, p is the number of parameters estimated in each model,
5. Find gradient from step 2 with every parameter in models, E.g. $\nabla\varphi_i$,
6. Substitute the initial value of the parameter (step 3) to gradient of step 5 with result. E.g. $\nabla\varphi_1$,
7. Find the value of p_1
8. Find the value of δ_1 so it will obtain value of $d_1$and s_1,
9. Substitute these values into the formula of L-BFGS-B optimization method;

10.Continue this step until parameter values converge to a value of 0.

4. Empirical results

In this study, empirical results use some data: data of Indonesian government bonds, Kurs, and JKSE. Data are taken from May 5 - 31, 2010. Data observation within month aims to look at correlation and collinearity among yields to macroeconomic factors such as Kurs and JKSE. Yield curves described in this study are described by-day with intention of making it easy to read changes that occur among yield whether to maturity or other factors that affect it. Results presented in this study use software R 2.15.5. The yield curves of various models are displayed using data of May, 7-10, 2010:

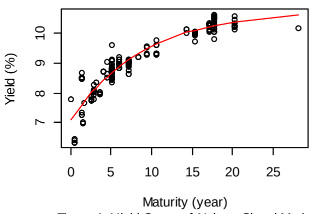

Figure 1. Yield Curve of Nelson-Siegel Model

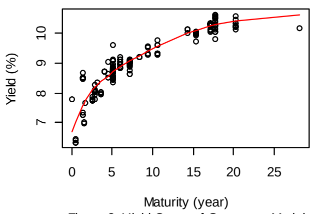

Figure 2. Yield Curve of Svensson Model

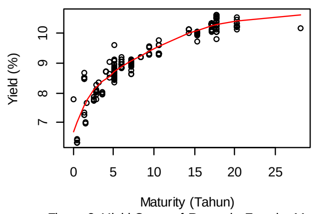

Figure 3. Yield Curve of Rezende-Ferreira Model

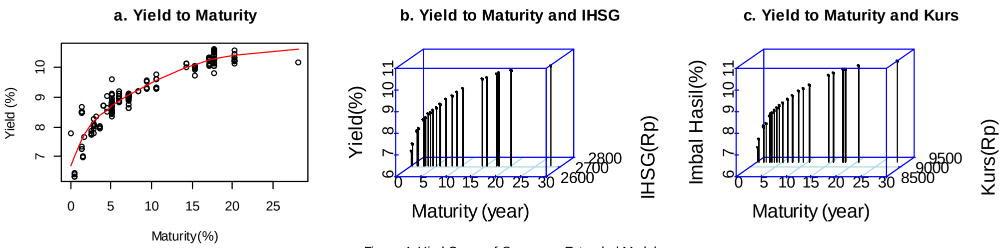

Figure 4. Yied Curve of Svensson Extended Model

To see the best of these model classes, MSE, RSME, and MAE are presented as follows:

Table 1. MSE, RMSE, and MAE of Nelson-Siegel Model Class

Model	MSE	RMSE	MAE
NS	0.07154131	0.03577066	52.03044
SV	0.06785191	0.03392595	50.34437
RF	0.06785191	0.03392596	50.34324
SV extended	0.06785191	0.03392596	50.34307

Based on table above, it can be concluded that MSE and RMSE for NS, SV, RF, and expansion of SV model have similarity, while MAE of four models is different. From difference, it is seen in that expansion of SV has smaller MAE. By taking smallest MAE, it can be concluded that the best model is extension of Svensson model.

5. Conclusions

The yield curve of Nelson-Siegel model and its extension consist of four models, i.e. Nelson-Siegel, Svensson, Ferreira-Rezende, and Svensson extended models. These four models are parametric models with linear and nonlinear parameters. These models have many local minimum values, so these models are more difficult to estimate. The estimation used in this model is NLS by L-BFGS-B method optimization approach. The estimation and optimization have a weakness in determination of initial parameter values. Determination of this value is difficult to reach global minimum. So determination of initial values of parameters of one data is not necessarily applicable to other data. This issue needs further study to find a suitable method to find initial parameter values as well as the best of estimation method for these models.

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