Tercentenary of *Ars Conjectandi* (1713)
Jacob Bernoulli and the Founding of Mathematical Probability

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Abstract

Jacob Bernoulli worked for many years on the manuscript of his book *Ars Conjectandi*, but it was incomplete when he died in 1705 at age 50. Only in 1713 was it published as he had left it. By then Pierre Rémond de Montmort had published his *Essay d'analyse sur les jeux de hazard* (1708), Jacob's nephew, Nicholas Bernoulli, had written a master's thesis on the use of the art of conjecture in law (1709), and Abraham De Moivre had published "De Mensura Sortis, seu de Probabilitate Eventuum in Ludis a Casu Fortuito Pendentibus" (1712). Nevertheless, *Ars Conjectandi* deserves to be considered the founding document of mathematical probability, for reasons explained in this paper.

By the "art of conjecturing" Bernoulli meant an approach by which one could choose more appropriate, safer, more carefully considered, and, in a word, more probable actions in matters in which complete certainty is impossible. He believed that his proof of a new fundamental theorem – later called the weak law of large numbers – showed that the mathematics of games of chance could be extended to a wide range civil, moral, and economic problems. Gottfried Wilhelm Leibniz boasted that Bernoulli had taken up the mathematics of probability at his urging. Abraham De Moivre pursued the project that Bernoulli had begun, at the same time shifting the central meaning of probability to relative frequency.

Key words: conjecture, Abraham De Moivre, G. W. Leibniz, weak law of large numbers.

> The origin of the frequentist theory of probability goes back to the question of whether one can compute the long-range frequency of some event E from known frequencies of some related events A, B, ... C. With an unavoidable degree of oversimplification, one might say that the theory of probability started in 1713, with the publication of the book *Ars Conjectandi*, by Jacob Bernoulli.
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> Jerzy Nyman (1976, 152)

1. Introduction and preliminaries.

Jacob Bernoulli worked for many years on his book *Ars Conjectandi*, but the manuscript was incomplete when he died in 1705 at age 50. In 1713 the Thurneysen Brothers in Basel published *Ars Conjectandi* as Bernoulli had left it in manuscript, together with his *Tractatus de seriebus infinitis* (five treatises on infinite series which had been printed separately between 1689 and 1704) and his *Lettre à un Amy, sur les Parties du Jeu de Paume*. In the years between Bernoulli's death and the publication of his book, his widow had retained control of the manuscript; his son Nicolaus Bernoulli had finally given the manuscript to the Thurneysen brothers to publish. Bernoulli's nephew, Nicolaus I Bernoulli (born in the same year as Jacob Bernoulli's son Nicolaus and given the Roman numeral I to distinguish him from other Nicolaus Bernoullis), had seen the work in progress when he was Jacob's student in the early 1700s, but Nicolaus I Bernoulli was traveling when the book was being prepared for publication and, contrary to what is frequently repeated, he did not edit it, but only wrote a two-page preface and checked for printing errors (Sylla 2006, 60 – 61). Nicolaus I wrote in his preface that the publishers might have hoped that Jacob's brother Johann, who was most suited to completing the work, or Nicolaus I himself would add to Jacob's book what was missing, but Johann was occupied by several other affairs and Nicolaus I was traveling and, when he returned, felt unequal to the job (Bernoulli 2006, 129 – 130). Johann himself said that the family refused to let him see the manuscript after he had left Basel in 1695 to take up the chair of mathematics in Groningen. Most likely Jacob's widow and son preferred to keep the work as Jacob had left it so that it would be clear that Jacob was the sole author.

By 1713 Pierre Rémond de Montmort had published his *Essay d'analyse sur les jeux de hazard* (1708), Jacob's nephew, Nicholas I Bernoulli, had written a master's thesis applying the art of conjecture to law (1709), and Abraham De Moivre had published "De Mensura Sortis, seu de Probabilitate Eventuum in Ludis a Casu Fortuito Pendentibus" (1712). Soon after Jacob Bernoulli's death in 1705, the éloges concerning him at the Académie Royale des Sciences in Paris and elsewhere had reported in (not always accurate) detail the contents of the manuscript of *Ars Conjectandi* (his former student Jacob Hermann had been given the responsibility of organizing Jacob Bernoulli's papers and preparing a description of his work for such éloges). The éloges helped to inspire the just mentioned publications of Montmort and De Moivre on games of chance. By 1713, Montmort was about to publish a greatly expanded edition of his *Essay d'analyse*, containing several letters from Nicholas I Bernoulli that reflect knowledge of the contents of Jacob's manuscript work, plus a single letter from Johann I Bernoulli. If *Ars Conjectandi* had never been published until the twentieth century, news of the contents of the manuscript would still have influenced the development of mathematical probability. After the work of Pascal, Fermat, and Huygens in the middle of the seventeenth century, there was almost no further advance in the field of reckoning in games of chance until the report of what Jacob Bernoulli had been working upon at the time of his death stimulated Montmort to publish his book. Steve Stigler calls the period between the 1650s and 1708 "the Dark Ages of Probability," as far as publication is concerned.

Although upstaged by the éloges, the actual publication of *Ars Conjectandi* in 1713 was important. Bernoulli's conception and proof of the fundamental theorem with which *Ars Conjectandi* ends (later called the weak law of large numbers) deserve close study. In Bernoulli's mind, the proof of his fundamental theorem showed that it would be possible to extend the mathematics of games of chance to a much larger group of naturally occurring situations including those involving civil, moral, and economic questions. Although others at the end of the seventeenth century were beginning to work on such problems as life expectancies or the pricing of annuities, they did not have a rationale to guide reasoning from experience to ratios of possible outcomes. Bernoulli himself had the rationale in his fundamental theorem, but he did not find as many good examples as he would have liked to exemplify the application of his new art of conjecturing to civil, moral, or economic problems. This helps to explain why the book was still in progress at the time of Bernoulli's death. In the years before his death, Bernoulli repeatedly asked Leibniz if he could loan him a copy of Jan de Witt's rare pamphlet in Dutch on annuities, *Waerdye van Lyf-Renten Naer proportie van Los-Renten* (1671), but Leibniz was unable to find de Witt's work, which he had reviewed, in the mess of his papers (Sylla 2006, 35, 45 – 49).

Jacob Bernoulli was the oldest of a family of mathematicians that has been called "the most renowned family in the history of the mathematical sciences" (Stigler 1986, 63). Starting with Jacob, members of the Bernoulli family held the chair of mathematics at Basel University continuously for more than a hundred years. Jacob Bernoulli (1654 – 1705) was professor of mathematics from 1687 until his death in 1705. He was followed by his younger brother Johann I Bernoulli (1667 – 1748), who held the chair from 1705 until his death in 1748, when he was followed by his son Johann II Bernoulli (1710 – 1790), who held the chair from 1748 to 1790.

The Roman numerals attached to the names Johann I and Johann II Bernoulli are part of the system already mentioned for Nicolaus I Bernoulli, which has been devised by historians to keep straight the various Bernoulli mathematicians with the same names. Nicolaus I Bernoulli (1687 – 1759), was the son of another brother of Jacob and Johann I Bernoulli, namely Nicolaus Bernoulli (1662 – 1716), who is called "the elder" by historians rather than being given a number because he was a portrait painter, and not a mathematician. Besides Johann II Bernoulli, who followed his father Johann I as professor of mathematics at Basel, Johann I Bernoulli had two other sons trained in mathematics who, for a time held positions in St. Petersburg: Nicolaus II Bernoulli (1695 – 1726), who died young shortly after moving to St. Petersburg, and Daniel Bernoulli (1700 – 1782), who is known for his work in hydrodynamics and for the St. Petersburg paradox in probability (Sylla 2006, 1 – 4). The label "Bernoulli's theorem" is usually understood to mean Daniel Bernoulli's theorem in fluid mechanics rather than Jacob Bernoulli's theorem in mathematical probability, leaving Jacob Bernoulli's fundamental theorem to be called the weak law of large numbers, a name which does not reflect Jacob Bernoulli's own conception of the theorem.

To further complicate the names, Jacob Bernoulli, author of *Ars Conjectandi*, also gave his own son the name Nicolaus. Jacob's son Nicolaus (1687 – 1769) became a painter and is called "the younger" to distinguish him both from his uncle, the painter Nicolaus "the elder," and his cousin Nicolaus I, born in the same year. This confusion of people named Nicolaus Bernoulli can be explained in part by the fact that both Jacob Bernoulli and his brother Nicolaus "the elder" were naming their sons after their father Nicolaus Bernoulli (1623 – 1708), a spice merchant. Jacob, Nicolaus "the elder", and Johann I had yet another brother, Hieronymus Bernoulli (1669 – 1760), who, like his father, was a *materialist* or seller of bulk spices, and who, because he was not a mathematician, is not usually mentioned in discussing the mathematical Bernoullis. Jacob Bernoulli's place in the Bernoulli family casts light on his motivations in developing the art of conjecturing, as will be described. The competition between the brothers Jacob and Johann I over their respective contributions to the development of the calculus of variations no doubt had repercussions on Jacob's progress in writing *Ars Conjectandi*. It is also worth considering what set of family values made it seem reasonable for Nicolaus I Bernoulli to make such extensive use of his uncle's unpublished manuscript in his own work on the application of the art of conjecturing in law. Some historians credit Nicolaus I with helping to promote the dissemination of Jacob Bernoulli's work in mathematical probability.

2. *Ars Conjectandi* as the founding document of mathematical probability.

Not appearing in print until 1713, Bernoulli's *Ars Conjectandi* might be thought to cede priority in the development of mathematical probability to the works of Montmort (1708), Nicolaus I Bernoulli (1709 and 1711), and De Moivre (1712), not to mention the 1654 correspondence of Blaise Pascal and Pierre de Fermat (first published in Fermat's *Varia Opera Mathematica*, 1679), Pascal's *Traité du triangle arithmétique* (written about 1654, but not released in print until 1665; Edwards 2002, 58), and Christiaan Huygens' *De ratiociniis in ludo aleae* (1657).

The standard history of the emergence of mathematical probability told by mathematicians begins with the correspondence between Pascal and Fermat in 1654 concerned in part with the problem of points, that is the division of the pay-out in a game that is stopped before its planned end. In mid-seventeenth-century Paris, Pascal and

Fermat had high reputations as mathematicians and their correspondence was well-known by word of mouth. Christiaan Huygens, when he visited Paris, heard of their letters and was motivated to write his own *De ratiociniis in ludo aleae* (*On reckoning in games of chance*). Although Huygens mentions the correspondence of Pascal and Fermat in *De ratiociniis* (and Bernoulli consequently includes this mention in his work) and Bernoulli mentions the printing of the correspondence in the *Opera* of Fermat, Bernoulli apparently had never seen Pascal's Treatise on the *Arithmetic Triangle* (Bernoulli 2006, 132, 157; Sylla 2006, 346).

To privilege the correspondence of Pascal and Fermat in 1654 in the history of mathematical probability, although it was not published at that time, is, in a way, to privilege the perspective of the French. This particular telling of the history has its primary origin in a history that Pierre Rémond de Montmort provided in the second edition of his *Essay d'analyse* as part of a claim for the importance of the 1708 edition of his work, which he felt that De Moivre had insufficiently appreciated in his 1712 publication (Sylla 2006). In favor of a history that gives less emphasis to Pascal and Fermat and more to *Ars Conjectandi* (and to Huygens' *De ratiociniis* with which it began), it might be noticed that up until the publication of *Ars Conjectandi*, what had existed were works on expectations in games of chance, where the chances of one outcome or another were known *a priori* based on the rules of the game or on the construction of the game pieces or cards, etc. New in *Ars Conjectandi* was an effort to deal with examples in which the ratios of possible outcomes were not known *a priori*, but had to be determined by past experience or *a posteriori*. Also new in *Ars Conjectandi* was the attempt to deal not only with games or lotteries and the like, but with civil, moral, and economic problems. Finally, the systematic deployment of the concept 'probability' in connection with such cases was also new. Probability, up to that time, had been an epistemic concept. It was assumed that in certain disciplines such as geometry one can demand proofs and certainty, but in other disciplines such as ethics certainty is unattainable. In the latter sorts of disciplines, the most one can hope for is probability, that is, a position that is reasonable, for which one can make persuasive arguments, or one that is approved by highly respected people. When people say that mathematical probability began with the correspondence of Pascal and Fermat in 1654, then, they are looking at the mathematics that Pascal and Fermat used and assimilating it to mathematics that would now be considered part of probability, even though Pascal and Fermat did not think in terms of the concept of probability that then existed.

In *Ars Conjectandi*, Bernoulli started from the mathematics of games of chance found in Huygens's *De ratiociniis*, enriched it by a systematic treatment of combinations and permutations, and then proposed to apply it, by analogy, to civil, moral, and economic questions. The epistemic probability of a statement, for instance that Cajus was guilty of committing a certain crime, was to be calculated by weighing and combining the arguments and evidence one way or another, using a mathematical expression like that used to calculate a person's expectation in a game. For Bernoulli, probability or degree of certainty primarily measured the likelihood that a given proposition or point of view is true. When Bernoulli proved his fundamental theorem, this framework was transformed by the insight that underlying ratios of possible outcomes could be investigated by *a posteriori* examination of outcomes in many similar cases in a way that also provided a measure of the probability that the answer obtained was correct within a given margin of error.

3. Is there a reason why Jacob Bernoulli did not finish writing *Ars Conjectandi*?

Stephen Stigler has suggested that Jacob Bernoulli did not publish *Ars Conjectandi* in his lifetime because he lacked an accepted standard that could tell him how close to certainty is “good enough” (Stigler 1999, 375). He suggests something similar happened with Thomas Bayes. It is true that Bernoulli advised that magistrates should decide how close to certainty a claim should be to be accepted as morally certain:

> It would be useful...if definite limits for moral certainty were established by the authority of the magistracy. For instance, it might be determined whether 99/100 of certainty suffices or whether 999/1000 is required. Then a judge would not be able to favor one side but would have a reference point to keep constantly in mind in pronouncing a judgment (Sylla 2006, 321).

Nevertheless, in one of the few examples existing in which Bernoulli applied his *a posteriori* method, namely in his *Lettre à un Amy sur les Parties du Jeu de Paume*, Bernoulli does not aim at high certainty, supposing that the relative strengths of two players be established by observing who wins 200 or 300 strokes.

How much longer might *Ars Conjectandi* have been had it been completed? Presumably, as in Parts I – III, after Bernoulli’s proof of his fundamental theorem, he would have wished to apply it in a number of examples, as he applied combinations and permutations to games in Part III. Part III is 71 pages long in the 1713 edition and slightly over 42 pages in the 1975 transcription; it includes 24 problems. Might Jacob Bernoulli have found as many as 24 significant problems related to civil, moral, and economic questions? In his 1709 masters’ thesis in law, which is slightly over 38 pages in the 1975 transcription, Nicolaus Bernoulli deals with life expectancy and annuities in some detail (about 8 pages and 11 pages respectively), and more briefly discusses how long a person should be missing before he is declared dead; restrictions on legacies; games, guarantees, and lotteries; the number of infants which a pregnant women will probably give birth to, and wills. Perhaps if Jacob Bernoulli had gotten hold of Jan de Witt’s pamphlet, he might have decided, like Nicolaus Bernoulli, that a thorough treatment of life expectancy and various types of annuities, with briefer attention to some other problems, would round out his book satisfactorily. As it was, the Thurneysen brothers compensated to some degree for the premature end to *Ars Conjectandi* by including in the same volume Bernoulli’s *Letter to a Friend on Parts in Court Tennis*, which provided an example of how ratios of strengths of players obtained *a posteriori* could be used in further calculations. The *Letter* exemplifies Bernoulli’s willingness to make use of ratios of possible outcomes obtained *a posteriori* with far less than the 1000 to 1 probability of accuracy within a small range that he had used as an example in his proof of his fundamental theorem.

4. Abraham De Moivre and the further development of mathematical probability

Abraham De Moivre and Jacob Bernoulli had somewhat similar backgrounds, although De Moivre was in England and Jacob Bernoulli in Basel. In an earlier generation the Bernoulli family had left the Catholic Netherlands as Protestant refugees. After staying for a while in Frankfurt, the family settled for good in Basel, Jacob’s grandfather becoming a citizen of Basel in 1622 through marriage. Abraham De Moivre, also a Protestant, fled France for England after the Revocation of the Edict of Nantes in 1685 prohibited the practice of the Protestant religion in France. In England, never able to find a position in a university, De Moivre supported himself by teaching mathematics to the children of the wealthy. After Isaac Newton left Cambridge University and moved to London to become Master of the Mint, Newton and De Moivre often met in Slaughter’s Coffee House in the evening for conversation (Bellhouse 2011, 28).

De Moivre appreciated *Ars Conjectandi*, but he thought that he could advance the mathematics it contained. When Nicolaus I Bernoulli wrote offering to send De Moivre a copy of the book, De Moivre replied that he already had one, and commented about Jacob’s *a posteriori* approach to ratios of cases (his proof of his fundamental theorem), “The problem of experiments [*experiences*] is of infinite beauty (Sylla 2006, xvi; “Le

Probleme des experiences est d'une beauté infinie.") In later editions of *The Doctrine of Chances*, De Moivre called Jacob and Nicolaus I Bernoulli "two treat Mathematicians," but said of their proofs of the fundamental theorem, "tho' they have shewn very great skill...yet some things were farther required; for what they have done is not so much an Approximation as the determining very wide limits, within which they demonstrated that the Sum of the Terms was contained" (Sylla 2006, xvii).

In adult life, neither De Moivre nor Jacob Bernoulli put a great emphasis on Christian religion (as was, for instance, the case for Blaise Pascal), but both of them referred to God in setting up the framework for mathematical probability. Jacob Bernoulli had been trained as a Protestant theologian and, following his father's wishes, had nearly embarked on a career as a Protestant clergyman, before having the good luck to obtain the chair of mathematics at the university in Basel. It was a shared assumption between De Moivre and Bernoulli and many others at the time that the events of the physical world are determined (except perhaps as they are affected by human free will) and that in any case God foresees the future with certainty. This meant that the art of conjecturing and mathematical probability were related only to human knowledge. They were tools to assist in decision making under conditions of uncertainty. They were not expected to provide scientific certainty.

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