The Extreme Value Birnbaum-Saunders Model in Athletics

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Abstract: The Birnbaum-Saunders (BS) model is a life distribution that has recently been largely studied and applied. A random variable following the BS distribution can be defined through a simple transformation of a standard normal. The BS model can thus be generalized by switching the standard normal distribution of the basis random variable, allowing the construction of more general classes of models. Among those models, we mention the extreme value Birnbaum-Saunders (EVBS) models, recently introduced in the literature, and based on results from extreme value theory. A real application to athletics data will be used to illustrate the methodology and to provide the way this model and related models can link with traditional extreme value analysis methods.

Key Words: Athletics, extreme value Birnbaum-Saunders models, parametric estimation, statistics of extremes.

1 Introduction and preliminaries

The *Birnbaum-Saunders* (BS) model is a life distribution, introduced and studied in Birnbaum and Saunders (1969), that has been largely applied in recent decades. BS and standard normal random variables (RVs), now denoted respectively by T and Z, are related by the formula

$$T = \delta\left(\alpha\, Z/2 + \sqrt{\{\alpha\, Z/2\}^2 + 1}\,\right)^2, \quad \text{i.e.} \quad Z = \left(\sqrt{T/\delta} - \sqrt{\delta/T}\right)/\alpha, \tag{1}$$

with $\alpha > 0$ and $\delta > 0$ shape and scale parameters, respectively. Consider the usual notations ϕ and Φ for the standard normal probability density function (PDF) and cumulative distribution function (CDF), respectively, and let

$$a_t = \left(\sqrt{t/\delta} - \sqrt{\delta/t}\right)/\alpha, \quad \text{so that} \quad a'_t = d\,a(t)/dt = \left(\sqrt{t/\delta} + \sqrt{\delta/t}\right)/(2\alpha t). \tag{2}$$

Then, the PDF and the CDF of T, in (1), are respectively $f_T(t) = a'_t\,\phi\big(a_t\big)$ and $F_T(t) = \Phi(a_t),\ \ t > 0$.

The assumption of a normally distributed Z can be obviously relaxed, assuming that Z follows any other distribution with PDF f_Z. We then obtain a general BS *type* (BST) RV, denoted by $T \sim \text{BST}(\delta, \alpha; f_Z)$, with a PDF $f_T(t) = a'_t\, f_Z(a_t),$ for $t > 0$, and a_t and a'_t as given in (2). Among those models, we mention the *extreme value Birnbaum-Saunders* (EVBS) case, recently introduced in Ferreira *et al.* (2012) and further considered in Gomes *et al.* (2012) and Ferreira (2013). These models are essentially based on results from *extreme value theory* (EVT).

In this article, after sketching the main limiting result on EVT, we introduce the EVBS models and make some comments on their importance. Finally, we provide an application to athletics data, showing the way some of these models can link with a traditional *extreme value analysis* (EVA) method, like the *block maxima* method.

2 The main limiting result in EVT and associated parametric model in statistics of univariate extremes

The main limiting results in EVT date back to the papers by Fréchet (1927), Fisher and Tippett (1928), von Mises (1936) and Gnedenko (1943), who fully characterized the possible non-degenerate limit laws, as $n \to \infty$, of the sequence of maximum values, $X_{n:n} := \max(X_1, \ldots, X_n)$, suitably normalized, with $X_1, \ldots, X_n$ independent, identically (IID) RVs from an underlying CDF F. More specifically, if there are normalizing constants $a_n > 0$, $b_n \in \mathbb{R}$ and some non-degenerate CDF, G, such that, for all $x \in \mathcal{C}(G)$, the set of continuity points of G, $\lim_{n\to\infty} P\big\{(X_{n:n} - b_n)/a_n \le x\big\} = G(x)$, we can redefine the constants in such a way that

$$G(x) \equiv G_\gamma(x) := \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \ 1+\gamma x > 0, & \text{if} \quad \gamma \neq 0, \\ \exp(-\exp(-x)), \ x \in \mathbb{R}, & \text{if} \quad \gamma = 0, \end{cases} \tag{3}$$

the so-called (general) *extreme value distribution* (EVD), given here in the von Mises-Jenkinson form (von Mises, 1936; Jenkinson, 1955), and denoted by $\mathrm{EV_M} \equiv \mathrm{EV_M}(\gamma)$. We then say that F is in the *max-domain of attraction* (MDA) of G_γ, in (3), and use the notation $F \in \mathcal{D}_{\mathrm{M}}(G_\gamma)$. The limiting CDFs, G, are then *max-stable* (MS), i.e., they are indeed the unique laws S such that the functional equation $S^n(\alpha_n x + \delta_n) = S(x)$, for $n \ge 1$, holds for some $\alpha_n > 0$ and $\delta_n \in \mathbb{R}$. The EVD reduces indeed to the Fréchet ($\gamma > 0$), Weibull ($\gamma < 0$) and Gumbel ($\gamma = 0$) CDFs, respectively. In fact, we often work with one of the three following types:

$$\begin{array}{lll} \text{Type I (Gumbel)}: & \Lambda(x) = \exp(-\exp(-x)), \ x \in \mathbb{R} & (\gamma = 0), \\ \text{Type II (Fréchet)}: & \Phi_\alpha(x) = \exp(-x^{-\alpha}), \ x \ge 0 & (\gamma = 1/\alpha), \\ \text{Type III (max-Weibull)}: & \Psi_\alpha(x) = \exp(-(-x)^{\alpha}), \ x \le 0 & (\gamma = -1/\alpha). \end{array} \tag{4}$$

The real parameter γ in (3), the primary parameter of interest in EVA, is the so-called *extreme value index* (EVI). The EVI rules the behaviour of the right-tail of F. If $\gamma < 0$, we have light right-tails, with a finite right endpoint, all in the so-called Weibull MDA. In addition, $\gamma = 0$ corresponds to the Gumbel MDA (exponential *right-tails*). And if $\gamma > 0$, we have the Fréchet MDA corresponding to heavy *right-tails* (polynomial tail decay, with an infinite right endpoint).

Remark 1. *All results developed for maxima can easily be reformulated for minima since* $X_{1:n} := \min\{X_1, \ldots, X_n\} = -\max\{-X_1, \ldots, -X_n\}$. *If we are interested in left-tails, we have, for the linearly normalised sequence of minimum values, a limiting* CDF, $G^*_{\gamma^*}(x) = 1 - G_{\gamma^*}(-x)$, *often referred to as a* $\mathrm{EV_m} \equiv \mathrm{EV_m}(\gamma^*)$ *models, i.e.,*

$$G^*_{\gamma^*}(x) = \begin{cases} 1 - \exp(-(1-\gamma^* x)^{-1/\gamma}), \ 1 - \gamma^* x > 0, & \textit{if} \quad \gamma^* \neq 0, \\ 1 - \exp(-\exp(x)), \ x \in \mathbb{R}, & \textit{if} \quad \gamma^* = 0. \end{cases} \tag{5}$$

The parameter γ^**, in* (5)*, determines the left-tail behavior of* F*, such as the parameter* γ*, in* (3)*, determines the right-tail behavior of* F*, being so both crucial parameters in* EVT.

From a statistical point of view, let us assume we have access to a sample $(X_1, \ldots, X_n)$ of n IID or possibly *stationary weakly dependent* RVs from an underlying CDF, F. Moreover, let us use the notation $(X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n})$ for the sample of associated ascending order statistics. *Statistics of univariate extremes* (SUE) help us to learn from disastrous or almost disatrous events, always of high relevance in society and with a high social impact. Its domains of application are thus quite diversified. We mention the fields of hydrology, meteorology, insurance, finance, telecommunications, athletics and biostatistics, among others (see, e.g., Reiss and Thomas, 2001, 2007). Although it is possible to find some historical papers with applications related to extreme events, the field dates back to Gumbel, in papers from 1935 on, summarized in his book (Gumbel, 1958). He developed statistical procedures essentially based on the aforementioned Gnedenko's limiting resullts. Indeed, parametric inference on the right-tail of F, usually unknown, is done on the basis of the approximation $P(X_{n:n} \le x) = F^n(x) \approx G_\gamma\left((x-\lambda_n)/\delta_n\right)$, with $(\lambda_n, \delta_n) \in (\mathbb{R}, \mathbb{R}^+)$ a vector of unknown location and scale parameters. The limiting result given before for the normalized sequence of maximum values validates such an approximation, and was used by Gumbel, to give approximations of this type but for any of the models in (4). He suggested the first model in *statistics of extremes*, usually called the *block maxima* model. Under this model, the sample of size n is divided into k sub-samples of size r (usually associated with k years, for $n = r \times k$, and r reasonably large). Next, the maximum of the r observations in each of the k sub-samples is considered, and one of the extremal models in (4), obviously with extra unknown location and scale parameters, is fitted to the sample of those k maximum values. Nowadays, whenever using this approach, still quite popular in environmental sciences, it is more common to fit to the data a univariate EVD, $G_\gamma((x-\lambda_r)/\delta_r)$, with G_γ given in (3), $(\lambda_r, \delta_r, \gamma_r) \in (\mathbb{R}, \mathbb{R}^+, \mathbb{R})$ unknown location, scale and 'shape' parameters. As mentioned above, we shall use the self-explanatory notation, $\mathrm{EV_M}(\lambda, \delta, \gamma)$, for such a type of models. We can analogously consider $\mathrm{EV_m}(\lambda, \delta, \gamma)$ models, whenever dealing with minimum values. All statistical inference is then related to the above mentioned models. Recent surveys on SUE can be found in Gomes *et al.* (2008), Beirlant *et al.* (2012) and McDonald and Scarrot (2012).

3 Some details on EVBS distributions

The $\mathrm{EVBS_M}$ (and $\mathrm{EVBS_m}$) distributions, based on limiting EV models for *maxima*, $\mathrm{EV_M}$, (and for *minima*, $\mathrm{EV_m}$), have been introduced in Ferreira *et al.* (2012). Specifically, consider that Z follows the EV distribution for maxima, in (3), i.e., $Z \sim \mathrm{EV_M}(\gamma)$. Then, we use the notation $\mathrm{EVBS_M}(\delta, \alpha, \gamma)$ for the CDF of the RV T, in (1). Analogously, if we consider that Z follows the EV distribution for *minima*, in (5), denoted by $Z \sim \mathrm{EV_m}(\gamma^*)$, and the same expression for T, i.e., that given in (1), we use the notation $T \sim \mathrm{EVBS_m}(\delta, \alpha, \gamma)$. Obviously, and if needed, we can further introduce a location λ, and work more generally with the RV $\lambda + T$, with $\lambda \in \mathbb{R}$. We shall then use the notations $\mathrm{EVBS_M}(\lambda, \delta, \alpha, \gamma)$ and $\mathrm{EVBS_m}(\lambda, \delta, \alpha, \gamma)$.

The EVBS models are very flexible, with extremely diversified left and right-tails. Moreover, the $\mathrm{EVBS_M}$ and $\mathrm{EVBS_m}$ *hazard rate* (HR) functions present *several different shapes* going through all possible HR shape classes, and this contrarily to the $\mathrm{EV_M}$ and $\mathrm{EV_m}$ HRs (see Ferreira *et al.*, 2012). These are very strong points in favour of $\mathrm{EVBS_M}$ and $\mathrm{EVBS_m}$ models, as they become quite rich and interesting for modeling purposes (see Ferreira *et al.*, 2012, and Gomes *et al.*, 2012).

Estimation aspects and model checking for EVBS distributions have been dealt with in Ferreira *et al.* (2012). The system of likelihood equations does not produce an explicit solution so that a numerical procedure is necessary. To analyze data from EVBS models, an **R** package named **evbs** is being developed, and its 'in progress' version is already available through the authors. This package contains diverse indicators, as well as methodologies useful for EVBS distributions, among which is the maximum likelihood (ML) estimation of the unknown parameters.

Once the EVBS distribution parameters have been estimated, a natural question is checking how good is the fit of the model to the data. In order to compare the EVBS distributions to other distributions, and just as in Ferreira *et al.* (2012) and Gomes *et al.* (2012), among others, we have used the following model selection criteria based on loss of information: *Akaike* (AIC), *Schwarz's Bayesian* (BIC) and *Hannan-Quinn* (HQIC) information criteria. These criteria are given by AIC $= -2\ell(\hat{\boldsymbol{\theta}}) + 2d$, BIC $= -2\ell(\hat{\boldsymbol{\theta}}) + d\log(n)$, and HQIC $= -2\ell(\hat{\boldsymbol{\theta}}) + 2d\log(\log(n))$, where $\ell(\hat{\boldsymbol{\theta}})$ is the *log-likelihood function* for the parameter $\boldsymbol{\theta}$ associated with the model evaluated at $\theta = \hat{\theta}$, n is the sample size, and d is the dimension of the parameter space. Generally, differences between two values of the aforementioned information criteria are not very noticeable. In that case, a suitable function of the Bayes factor (BF) B_{12}, i.e. BF$^* := 2\log(B_{12})$, can be used to highlight such differences, if they exist. If BF$^* < 0$, evidence in favor of model M_1 is negative and M_2 should be accepted. If BF$^* \in [0, 2)$, there is a week (W) evidence in favor of M_1. Values of BF* in $[2, 6)$, $[6, 10)$ or $[10, \infty)$ provide respectively positive (P), strong (S) or very strong (VS) evidence in favor of M_1 (see Vilca *et al.*, 2011, for details). AIC, BIC and HQIC are based on a penalization of the likelihood function as the model becomes more complex, i.e., with more parameters (Sanhueza *et al.*, 2008). Since models with more parameters should provide a better fit, AIC, BIC and HQIC allow us to compare models with different numbers of parameters due to the penalization incorporated in such criteria.

4 An application to athletics

In this section, we shall be interested in an application of $\mathrm{EVBS_M}$ and $\mathrm{EV_M}$ models to the best personal marks attained at a few athletic events, in a context similar to the one used in Henriques-Rodrigues *et al.* (2011), among others. We shall pay special attention to the estimation of γ, in (3). The data, already analysed in Henriques-Rodrigues *et al.* (2011), are related to two jumping events, all for men, the high jump (HJ) and the pole vault (PV). The *sources* were http://www.iaaf.org/statistics/toplists/index.htmx and http://hem.bredband.net/athletics/athletics_all-time_best.htm. Data were collected until the end of 2007 and for any athlete only the best mark was taken into account.

Due to the fact that the observed data considered are already maxima, possibly of a small and dependent number of marks associated with any of the n athletes, but the EV limiting law, in (3), is "robust" to changes of the IID assumption, we have tried the fitting, through ML, of an extreme value model $\mathrm{EV_M}(\lambda, \delta, \gamma)$, comparatively with the fitting of a $\mathrm{EVBS_M}(\lambda, \delta, \alpha, \gamma)$. For the ML estimation of the unknown parameters $(\lambda, \delta, \gamma)$ in the $\mathrm{EV_M}(\lambda, \delta, \gamma)$ model, we have used the **EVIR** package in the **R**-software. On the basis of the aforementioned **evbs R** package, we have proceeded with the ML estimation of $(\lambda, \delta, \alpha, \gamma)$, in the $\mathrm{EVBS_M}(\lambda, \delta, \alpha, \gamma)$ model. Table 1 presents the ML estimates of $(\lambda, \delta, \gamma)$ for the EV model, and of $(\lambda, \delta, \alpha, \gamma)$ for the EVBS models. In Table 2, we present the respective AIC, BIC, HQIC and BF* indicators.

For the HJ and PV data sets, the smallest value of $-\ell$ favours the EVSB model (see Table 1). The same happens with the AIC, BIC and HQIC (see Table 2). Also, BF* provides a strong evidence in favor of the EVBS model for the HJ data, and a very strong evidence for the PV data. The quality of the fitting, i.e., the good coherence between empirical and theoretical densities and distributions, can be visualised in Figure 1.

Table 1: ML estimates for the indicated models and data sets.

	Model	$\widehat{\lambda}$	$\widehat{\delta}$	$\widehat{\alpha}$	$\widehat{\gamma}$	$-\ell$
HJ	EVBS	2.19271	0.04014	0.85953	-0.37179	**-417.7767**
	EV	2.24045	0.03341	–	-0.09084	-414.6809
PV	EVBS	5.42133	0.16522	0.70185	-0.40329	**-151.5884**
	EV	5.57959	0.09477	–	-0.15096	-141.2343

Table 2: AIC, BIC, HQIC and BF* for the indicated models and data sets.

	Model	AIC	BIC	HQIC	BF*
HJ	EVBS	**-827.5534**	**-814.2614**	**-822.1771**	—
	EV	-827.3618	-813.3928	-819.3296	6.1916 (S)
PV	EVBS	**-295.4686**	**-261.176**	**-289.8005**	—
	EV	-274.468	-0.64999	-269.0923	20.7082 (VS)

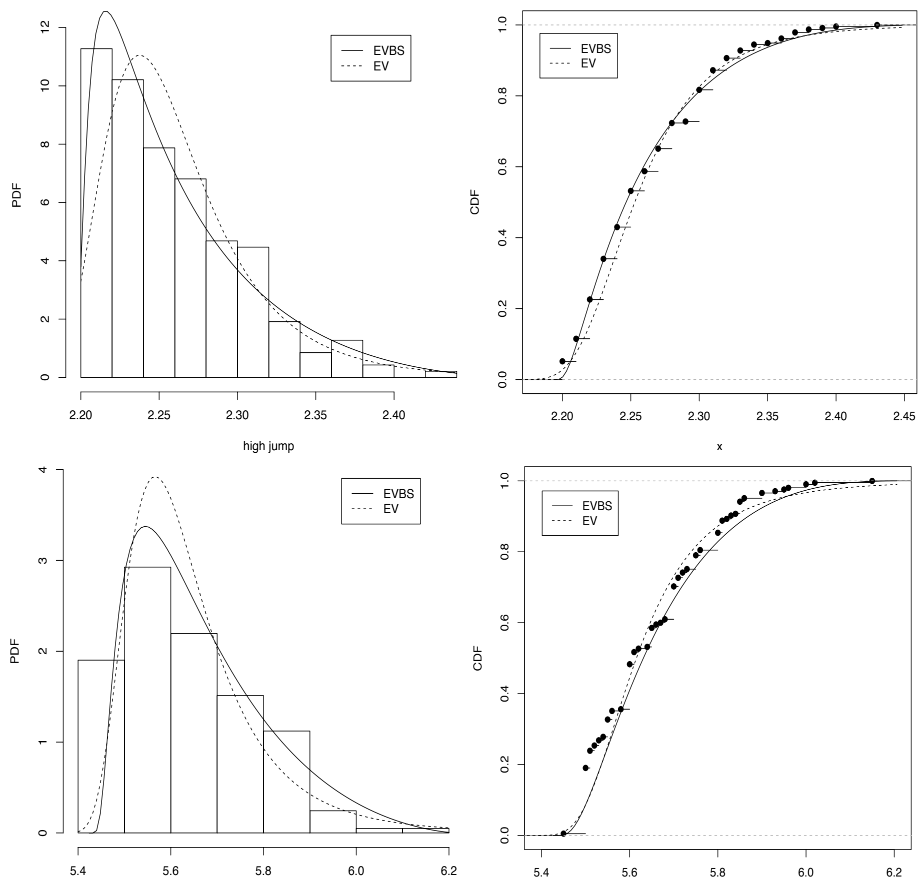

Figure 1: HJ Histogram (*top, left*) and CDF (*top, right*); PV Histogram (*bottom, left*) and CDF (*bottom, right*).

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